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Reversible interval homeomorphisms

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Abstract

A characterization of continuously reversible self-mappings of an open interval, that is, compositions of two continuous involutions, is proved.

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1. Introduction

The notion of reversibility has a physical origin and has been studied mainly for multidimensional smooth dynamical systems (see, e.g., [1] and [5]). It precises mathematically the invariance of dynamics of a process under reversing the direction of time. This means that the function f describing the process and its inverse f^{-1} are conjugated by a function α which is the inverse of itself, i.e., an *involution*:

$$\alpha \circ f = f^{-1} \circ \alpha. \tag{1}$$

Then $f^{-1} = \alpha \circ f \circ \alpha$ whence, by induction,

$$f^{-n} = \alpha \circ f^n \circ \alpha \quad \text{for } n \in \mathbb{Z}$$

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which is the desired invariance. Since usually we are interested in the context of continuity we come to the following precise definition.

A homeomorphism f of a topological space X is said to be *continuously reversible* if there exists a continuous involution $\alpha : X \to X$ such that (1) holds.

In [2] it was proved that the class of continuously reversible homeomorphisms of a real interval is pretty large. In particular, any homeomorphism of an open interval is a composition of two continuously reversible ones (cf. [2, Theorem 1 and Proposition 2]). The aim of the present paper is to give a full and quite simple description of all continuously reversible interval homeomorphisms (cf. Section 4 below).

2. Preliminaries on reversibility

There is another, less dynamical, definition of reversibility. The following fact states that both of them are equivalent.

Property 1 (cf. [2, Remark 1]). *A homeomorphism of a topological space is continuously reversible if and only if it is a composition of two continuous involutions.*

The next statement is quite clear.

Property 2. *Every continuous involution is continuously reversible.*

From now on we confine ourselves to self-mappings of sets of reals. A description of decreasing homeomorphisms which are continuously reversible is trivial.

Property 3 [2, Remark 4]. *A decreasing homeomorphism of an interval is continuously reversible if and only if it is an involution.*

For increasing homeomorphisms the situation is much more complicated. The proof of the following statement is quite elementary but is not obvious.

Property 4 [2, Theorem 1]. *Every homeomorphism of an interval having no fixed points is continuously reversible*; *more exactly*: *there exists a continuous decreasing involution conjugating the homeomorphism and its inverse function.*

There do exist interval homeomorphisms which are not continuously reversible (see Examples 3 and 4 and Theorem 1, also [2, Example 1]). In Theorem 1 we will give a characterization of increasing homeomorphisms which are continuously reversible.

3. Continuity and monotonicity of real involutions

Among fundamental properties of involutions the following two are important for us.

Property 5 ([3, Theorem 15.2], also [2, Remark 3]). *The only increasing involution defined on a set of reals is the identity function.*

Property 6 [3, Theorem 15.3]. *An involution defined on an interval is continuous if and only if it is monotonic.*

As follows from Examples 1 and 2 below this equivalence is not still true if the domain of an involution is not an interval.

Example 1. The function $\alpha : [0, 1/2) \cup [3/4, 1] \to [0, 1/2) \cup [3/4, 1]$, given by

$$\alpha(x) = \begin{cases} 1 - x, & \text{if } x \in [0, 1/4] \cup [3/4, 1], \\ 3/4 - x, & \text{if } x \in (1/4, 1/2), \end{cases}$$

is a decreasing involution which is discontinuous at $1/4$.

Example 2. The formula

$$\alpha(x) = \begin{cases} 1/2 - x, & \text{if } x \in (0, 1/2), \\ 3/2 - x, & \text{if } x \in (1/2, 1), \end{cases}$$

defines a continuous involution $\alpha : (0, 1) \setminus \{1/2\} \to (0, 1) \setminus \{1/2\}$ which is not monotonic.

We have, however, the following positive result.

Lemma 1. *Let F be a set which is closed in the topology of an interval and let $\alpha : F \to F$ be an involution. If α is monotonic then it is continuous.*

Proof. Assume that F is closed in an interval X. By Property 5 we may confine ourselves to the case when α is decreasing.

Fix an $x_0 \in F$. We will prove that α is left continuous at x_0. Take any increasing sequence $(x_n\colon n \in \mathbb{N})$ of points of F converging to x_0. Then

$$\alpha(x_n) \geqslant \alpha(x_{n+1}) \geqslant \alpha(x_0) \quad \text{for } n \in \mathbb{N},$$

so $(\alpha(x_n)\colon n \in \mathbb{N})$ converges to a $y_0 \in \mathbb{R}$. Clearly,

$$\alpha(x_n) \geqslant y_0 \geqslant \alpha(x_0) \quad \text{for } n \in \mathbb{N}$$

and, consequently, $y_0 \in X$. Therefore $y_0 \in F$, whence

$$x_n \leqslant \alpha(y_0) \leqslant x_0 \quad \text{for } n \in \mathbb{N}.$$

Thus, since $\lim_{n\to\infty} x_n = x_0$, we have $\alpha(y_0) = x_0$, i.e., $y_0 = \alpha(x_0)$. This means that $(\alpha(x_n)\colon n \in \mathbb{N})$ converges to $\alpha(x_0)$. □

4. Reversibility and symmetricity

In this section we are going to characterize continuously reversible homeomorphisms among increasing ones. Fix an open interval X.

Let F be a non-void set which is closed in the topology of X and let $\alpha : F \to F$ be a decreasing involution. Two components of $X \setminus F$ are called *α-symmetric* if α sends the left endpoint of one of them to the right endpoint of the other. It is clear that for every component I of $X \setminus F$ there is exactly one component J such that I and J are α-symmetric; more exactly: if I is of the form (p,q), $X \cap (-\infty, q)$, or $X \cap (p, \infty)$ then J equals $(\alpha(q), \alpha(p))$, $X \cap (\alpha(q), \infty)$, or $X \cap (-\infty, \alpha(p))$, respectively.

A homeomorphism $f : X \to X$ is said to be *symmetric* if either the set $\operatorname{Fix} f$ of all fixed points of f is empty, or there is a decreasing involution $\alpha : \operatorname{Fix} f \to \operatorname{Fix} f$ such that on α-symmetric components of $X \setminus \operatorname{Fix} f$ the graph of f lies on the same side of the diagonal.

If $f : X \to X$ is a decreasing homeomorphism then $\operatorname{Fix} f$ is a singleton and the graph of f lies over the diagonal on exactly one (viz. left) of the two components of $X \setminus \operatorname{Fix} f$. Consequently, such an f cannot be symmetric. Thus we have what follows.

Remark 1. Every homeomorphism with no fixed points is symmetric and every symmetric homeomorphism is increasing.

Below are examples of increasing homeomorphisms which are not symmetric.

Example 3. If $f : (0, 4\pi) \to (0, 4\pi)$ is given by

$$f(x) = \begin{cases} x + \sin x, & \text{if } x \in (0, 4\pi) \setminus (\pi, 2\pi), \\ x, & \text{if } x \in (\pi, 2\pi), \end{cases}$$

then $\operatorname{Fix} f = [\pi, 2\pi] \cup \{3\pi\}$ and there is no decreasing involution $\alpha : \operatorname{Fix} f \to \operatorname{Fix} f$ at all.

Example 4. Define $f : (0, 2\pi) \to (0, 2\pi)$ by $f(x) = x + \sin x$. Then $\operatorname{Fix} f = \{\pi\}$ and there is exactly one decreasing involution $\alpha : \operatorname{Fix} f \to \operatorname{Fix} f$ (viz. the identity function). The components $(0, \pi)$ and $(\pi, 2\pi)$ are α-symmetric but the graph of f on them lies on different sides of the diagonal.

Now we are in position to formulate the main result of the paper.

Theorem 1. *An increasing homeomorphism of an open interval is continuously reversible if and only if it is symmetric.*

In the proof the following lemma will be useful.

Lemma 2. *Let I and J be disjoint open intervals and let f be an increasing homeomorphism mapping $I \cup J$ onto $I \cup J$ and such that*

$$f(x) < x \quad (f(x) > x) \quad \textit{for } x \in I \cup J.$$

Then for every points $x_0 \in I$, $y_0 \in J$ and for every strictly decreasing function α_0 mapping $(f(x_0), x_0]$ onto $[y_0, f^{-1}(y_0))$ (mapping $[x_0, f(x_0))$ onto $(f^{-1}(y_0), y_0]$) there exists exactly one decreasing involution $\alpha : I \cup J \to I \cup J$ extending α_0 and satisfying (1); *moreover, α is continuous and $\alpha(I) = J$.*

Proof. Fix points $x_0 \in I$, $y_0 \in J$ and strictly decreasing function α_0 mapping $(f(x_0), x_0]$ onto $[y_0, f^{-1}(y_0))$. Since such a function is continuous it follows from [3, Lemma 3.1] (see also [4, Theorem 5.3.1]) that there exists a unique continuous $\alpha_1 : I \to J$ extending α_0 to a solution of (1); moreover, it may be immediately verified that α_1 is strictly decreasing and maps I onto J. Then the formula

$$\alpha(x) = \begin{cases} \alpha_1(x), & \text{if } x \in I, \\ \alpha_1^{-1}(x), & \text{if } x \in J, \end{cases}$$

defines a strictly decreasing involution mapping $I \cup J$ onto itself and such that $\alpha(I) = J$. If $x \in I$ then $f(x) \in I$, whence

$$\alpha(f(x)) = \alpha_1(f(x)) = f^{-1}(\alpha_1(x)) = f^{-1}(\alpha(x));$$

if $x \in J$ then $y := f^{-1}(\alpha_1^{-1}(x)) \in I$, so

$$\alpha_1(f(y)) = x,$$

that is, by (1),

$$f^{-1}(\alpha_1(y)) = x,$$

whence

$$\alpha(f(x)) = \alpha_1^{-1}(f(x)) = y = f^{-1}\big(\alpha_1^{-1}(x)\big) = f^{-1}(\alpha(x)).$$

Consequently, α satisfies (1). Its uniqueness follows immediately from the uniqueness of α_1. □

Finally, we come back to Theorem 1. In the proof below given sets $A, B \subset \mathbb{R}$ we write $A < B$ ($A \leqslant B$) iff $a < b$ ($a \leqslant b$) for every $a \in A$ and $b \in B$.

Proof of Theorem 1. Fix an open interval X and an increasing homeomorphism $f : X \to X$. By Property 4 we may assume that $\operatorname{Fix} f \neq \emptyset$ and $\operatorname{Fix} f \neq X$.

First assume that f is continuously reversible. Let $\alpha : X \to X$ be a decreasing involutory solution of (1). Then $\alpha(\mathrm{Fix}\, f) \subset \mathrm{Fix}\, f$, so $\alpha|_{\mathrm{Fix}\, f}$ is a decreasing involution. Fix a component of $X \setminus \mathrm{Fix}\, f$. Then $\alpha(I)$ is $\alpha|_{\mathrm{Fix}\, f}$-symmetric to I. Assume, for instance, that

$$f(x) < x \quad \text{for } x \in I.$$

If $y \in \alpha(I)$ then $\alpha(y) \in I$, so $f(\alpha(y)) < \alpha(y)$ whence, since α is strictly decreasing, we have by (1)

$$f^{-1}(y) = \alpha \circ f \circ \alpha(y) > \alpha(\alpha(y)) = y,$$

that is, $f(y) < y$. Consequently,

$$f(y) < y \quad \text{for } y \in \alpha(I)$$

which was to be proved.

Now assume that f is symmetric and let $\alpha_0 : \mathrm{Fix}\, f \to \mathrm{Fix}\, f$ be a decreasing involution such that on α_0-symmetric components of $X \setminus \mathrm{Fix}\, f$ the graph of f lies on the same side of the diagonal. Since α_0 is a decreasing involution the sets

$$F_- := \{x \in \mathrm{Fix}\, f \colon \alpha_0(x) \geqslant x\} \quad \text{and} \quad F_+ := \{y \in \mathrm{Fix}\, f \colon \alpha_0(y) \leqslant y\}$$

are non-void. If $x > y$ for some $x \in F_-$ and $y \in F_+$ we would have

$$x \leqslant \alpha_0(x) < \alpha_0(y) \leqslant y.$$

Therefore $F_- \leqslant F_+$. Let $\xi_- := \sup F_-$ and $\xi_+ := \inf F_+$. Then $\xi_- \leqslant \xi_+$. It follows from Lemma 1 that α_0 is continuous. Thus

$$\alpha_0(\xi_-) \geqslant \xi_- \quad \text{and} \quad \alpha_0(\xi_+) \leqslant \xi_+. \tag{2}$$

Consider the case $\xi_- = \xi_+$. Putting $\xi_0 := \xi_-$ we have $\alpha_0(\xi_0) = \xi_0$. Clearly, α_0 generates a one-to-one function A_0 mapping the family $\mathcal{I}_0$ of all the components of $X \setminus \mathrm{Fix}\, f$ contained in (ξ_0, ∞) onto the family of the components contained in $(-\infty, \xi_0)$. If $I \in \mathcal{I}_0$ then the components I and $A_0(I)$ are α_0-symmetric. Moreover,

$$I < J \quad \text{iff} \quad A_0(J) < A_0(I) \quad \text{for } I, J \in \mathcal{I}_0.$$

Therefore, using Lemma 2, for every component $I \in \mathcal{I}_0$ we can construct a decreasing continuous involution $\alpha_I : I \cup A_0(I) \to I \cup A_0(I)$ satisfying (1) and such that $\alpha_I(I) = A_0(I)$. Consequently, the function $\alpha : X \to X$, given by

$$\alpha(x) := \begin{cases} \alpha_0(x), & \text{if } x \in \mathrm{Fix}\, f, \\ \alpha_I(x), & \text{if } x \in I \cup A_0(I) \text{ and } I \in \mathcal{I}_0, \end{cases}$$

is a decreasing continuous involution satisfying (1), that is, f is continuously reversible.

Finally, assume that $\xi_- < \xi_+$. Then $F_- < F_+$ and, consequently, α_0 has no fixed points. Therefore, by (2),

$$\alpha_0(\xi_-) > \xi_- \quad \text{and} \quad \alpha_0(\xi_+) < \xi_+. \tag{3}$$

Moreover,

$$(\xi_-, \xi_+) \cap \operatorname{Fix} f = \emptyset \tag{4}$$

and, also by (3),

$$\alpha_0(\xi_-) = \xi_+ \quad \text{and} \quad \alpha_0(\xi_+) = \xi_-. \tag{5}$$

According to (4) and Property 4 there exists a continuous decreasing involution $\alpha_{00} : (\xi_-, \xi_+) \to (\xi_-, \xi_+)$ satisfying (1). Then we proceed similarly as in the case when $\xi_- = \xi_+$. It follows from (5) that α_0 generates a one-to-one function A_+ mapping the family $\mathcal{I}_+$ of all the components of $X \setminus \operatorname{Fix} f$ contained in (ξ_+, ∞) onto the family of the components contained in $(-\infty, \xi_-)$. Like in the previous case, applying Lemma 2, for every component $I \in \mathcal{I}_+$ we can find a decreasing continuous involution $\alpha_I : I \cup A_+(I) \to I \cup A_+(I)$ satisfying (1) and such that $\alpha_I(I) = A_+(I)$. The function $\alpha : X \to X$, defined by

$$\alpha(x) := \begin{cases} \alpha_0(x), & \text{if } x \in \operatorname{Fix} f, \\ \alpha_{00}(x), & \text{if } x \in (\xi_-, \xi_+), \\ \alpha_I(x), & \text{if } x \in I \cup A_+(I) \text{ and } I \in \mathcal{I}_+, \end{cases}$$

is a desired decreasing continuous involution satisfying (1). This completes the proof of continuous reversibility of f. $\square$

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References

[1] R.L. Devaney, Reversible diffeomorphisms and flows, Trans. Amer. Math. Soc. 218 (1976) 89–113.

[2] W. Jarczyk, Reversibility of interval homeomorphisms with no fixed points, Aequationes Math. 63 (2002) 66–75.

[3] M. Kuczma, Functional Equations in a Single Variable, in: Monografie Mat., Vol. 46, Polish Scientific Publishers, Warszawa, 1968.

[4] M. Kuczma, B. Choczewski, R. Ger, Iterative Functional Equations, in: Encyclopedia of Mathematics Its Applications, Vol. 32, Cambridge University Press, Cambridge, 1990.

[5] M.B. Sevryuk, Reversible Systems, in: Lecture Notes in Math., Vol. 1211, Springer-Verlag, Berlin, 1986.